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# LOGISTIC MODEL FOR STOCK MARKET BUBBLES AND ANTI-BUBBLES

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Log-periodic power laws often occur as signatures of impending criticality of hierarchical systems in the physical sciences. It has been proposed that similar signatures may be apparent in the price evolution of financial markets as bubbles and the associated crashes develop. The features of such market bubbles have been extensively studied over the past 20 years, and models derived from an initial discrete scale invariance assumption have been developed and tested against the wealth of financial data with varying degrees of success. In this paper, the equations that form the basis for the standard log-periodic power law model and its higher extensions are compared to a logistic model derived from the solution of the Schröder equation for the renormalization group with nonlinear scaling function. Results for the S&P 500 and Nikkei 225 indices studied previously in the literature are presented and compared to established models, including a discussion of the apparent frequency shifting observed in the S&P 500 index in the 1980s. In the particular case of the Nikkei 225 anti-bubble between 1990 and 2003, the logistic model appears to provide a better description of the large-scale observed features over the whole 13-year period, particularly near the end of the anti-bubble.

*Keywords:* Log-periodic power laws; renormalization group; financial bubbles; discrete scale invariance; criticality.

## 1. Introduction

The work of Sornette *et al.* (1996) and Feigenbaum & Freund (1996), which spawned the ubiquitous log-periodic power-law (LPPL) models for bubble formation in financial markets, had their origins in phenomenological observations of the Standard & Poor's (S&P) 500 index in the period leading up to the crash of October 1987 and a similar time series progression in the Dow Jones index preceding the great crash of 1929.

In this paper, we take a step back from the evolution of these ideas over the past 20 years and revisit the initial renormalization group formalism from which a series of models have been derived to fit the observed data. In the spirit of the

work of Curtright & Zachos (2011) in the physical context, we suggest a new series of models derived from the logistic differential equation which gives a nonlinear perturbation of the standard linear transformation in the renormalization group equation.

Each of these models, ours included, can be thought of as an attempt to describe the periodic fluctuations around a faster than exponential development in prices observed in financial time series prior to many of the significant market collapses. If a constant-return hypothesis is assumed, one expects to see exponential price development, but, during periods in which the market price is in an accelerating regime prior to a large drawdown, price development can be observed to follow a power law. It has long been suggested (see, for example, Sornette *et al.* (1996)) that this unsustainable growth up until some critical time is a consequence of the tendency for traders or market professionals to exhibit imitative or cooperative behavior. Furthermore, if analogies of studies of hierarchical systems in the natural sciences (Sornette 2002, 2009) are to be extended to financial markets, it would be reasonable to expect observations of discrete scale invariance around this critical time (Feigenbaum & Freund 1996), implying log-periodic corrections to the power law, which are indeed observed in some instances. This fundamental expectation has been the genesis of the last twenty years of discussion on the matter of financial crashes as critical points (Geraskin & Fantazzini 2013).

The original LPPL model (which we refer to here as the first-order model) was extended in Sornette & Johansen (1997) to take account of an apparent frequency shift in the log-periodic oscillation observed in the S&P 500 index from the beginning of 1980 to the crash of 1987. This was achieved by considering a Landau expansion and the model was further extended in Johansen & Sornette (1999a) by considering the next order of this expansion. This completes what we refer to as the original LPPL hierarchy of models.

It has been suggested in Vandewalle *et al.* (1998a) that logarithmic divergence with a universal critical exponent of zero could be preferable to the power law divergence exhibited by the above hierarchy (although Johansen & Sornette (1999b) note this provides inferior calibration results), and there have been other variations presented in the literature. However, in the considerations below, we confine discussion to the principal three LPPL models as comparators of the new models presented in this paper.

These models, their relevance and their predictive powers have not been without their detractors, see Laloux *et al.* (1998), Feigenbaum (2001), Sornette *et al.* (2013), and Bree & Joseph (2013) amongst others. However, in this paper we do not concern ourselves with these controversies, but rather present what we hope is a useful addition to the available toolbox of models that are based on the renormalization group and that go some way, at least, to explaining how financial bubbles develop.

This paper is structured as follows: we begin Sec. 2 with some revision on the motivations and derivation behind the current LPPL models primarily developed by Sornette and co-workers, followed by the derivation of a new model (and its extension) derived from the logistic differential equation. This can be thought of as a nonlinear perturbation of the renormalization group models map from which the LPPL models have been derived. In Sec. 3, we compare how well these logistic models describe the S&P 500 bubble preceding the stock market crash of 1987 and the long Nikkei 225 index bear market between 1990 and 2003, in comparison with the hierarchy of LPPL models, and we discuss the fitting methodology used for the logistic models. We conclude in Sec. 4 with suggestions for further investigations.

## 2. Renormalization Group Model with Linear and Nonlinear Scaling Corrections

In this section, we outline the LPPL model hierarchy and derive the logistic model and its extension. Our starting point is the renormalization group equation itself (which is used to derive models for direct fitting to either price or log-price time series), rather than the stochastic differential equation with specified hazard rate derived from the renormalization group equation (Johansen *et al.* 2000). This latter has been the focus of later work in the field and has become known as the Johansen–Ledoit–Sornette model of rational expectation bubbles with finite-time singularity hazard rate (Sornette *et al.* 2013).

Therefore, for our purposes, we introduce the following notation, which follows the notation used by Sornette and co-workers. Let  $I(t)$  be a stock index at time  $t$ . In practice,  $I(t)$  is frequently taken to be the logarithm of a stock index, thereby linearizing any constant rate of return trend, but the theory we give here applies equally to both cases. The critical time is denoted  $t_c$  and the time to criticality by  $x$ . If we are considering an asset bubble (for which the critical time is in the future and  $I(t)$  is trending upwards), we write  $x = t_c - t$  and write  $F(x) = I(t_c) - I(t)$ . For an anti-bubble (for which the critical time is in the past and  $I(t)$  is trending downwards) we write  $x = t - t_c$  and  $F(x) = I(t_c) - I(t)$ . This enables us to write the renormalization group equations in a standard form which is applicable to both bubbles and anti-bubbles. Note that the critical time corresponds to  $x = 0$  in both cases, and that  $x = |t_c - t|$ . We further note that for bubbles  $t_c > t$  and similarly for anti-bubbles  $t_c < t$ .

The derivation of the basic power-law model of a stock index  $I(t)$  at time  $t$  is founded on three assumptions: (i) there is some critical time  $t = t_c$  at which point  $I(t)$  is singular in its derivative and  $I(t_c) \neq 0, \infty$ ; (ii) the value of the index at time  $t$  is related to the value of the index at some other time, which in the  $x$  coordinate is denoted  $\phi(x)$ , and; (iii) at  $t = t_c$ ,  $F(x)$  is invariant with some appropriate scaling, under the transformation  $\phi(x)$ , i.e.  $F(x)$  exhibits nonlinear discrete scale invariance.

This leads to the renormalization-group approach to financial bubbles, for which close to the critical time  $x = 0$ , the real function  $F(x)$  satisfies the renormalization group equation<sup>a</sup>:

$$F(x) = \mu^{-1}F(\phi(x)), \quad \phi(x) = \lambda x + O(x^2), \quad x > 0, \quad (2.1)$$

where  $\lambda$  and  $\mu$  are positive constants. The function  $F(x)$  satisfies  $F(0) = 0$  with  $F'(x)$  singular at  $x = 0$ . The solution set of (2.1) depends significantly on the smoothness class of  $\phi(x)$  and  $F(x)$ . In relation to stock-market indices it is usual to assume that  $\phi(x)$  is a differentiable function near to  $x = 0$ , with  $\phi(0) = 0$ ,  $\phi'(0) = \lambda > 0$ , and that  $F$  is continuous at  $x = 0$ .

## 2.1. Log-periodic power-law model hierarchy

### 2.1.1. Linear scaling

Suppose  $\phi$  is linear so that  $\phi(x) = \lambda x$ . Then, as can be readily verified, the general solution of (2.1) for  $F(x)$  can be written as

$$F(x) = x^\beta G(\omega \log x), \quad x > 0, \quad (2.2)$$

where  $\beta = \log \mu / \log \lambda$ ,  $\omega = 2\pi / \log \lambda$  and  $G(x)$  is any function satisfying  $G(x) = G(x + 2\pi)$ .

Another approach is to complexify  $F$  and to seek a solution of the form  $F(x) = Ax^\alpha$ , where  $A, \alpha \in \mathbb{C}$ . When  $\alpha \in \mathbb{R}$  we obtain the relation  $\alpha = \beta = \log \mu / \log \lambda$  and the power-law model

$$I(t) = A + B|t_c - t|^\beta, \quad (2.3)$$

where  $A = I(t_c)$  and  $B$  are constants. Typically, for financial time series,  $\mu < \lambda$  so that  $0 < \beta = \log \mu / \log \lambda < 1$ , which corresponds to continuity of  $I(t)$  at  $t_c$  but a discontinuity in the first derivative of  $I(t)$  at  $t_c$ .

The expectation of discrete scale invariance around the critical time requires  $\alpha \in \mathbb{C}$  and  $\lambda^\alpha / \mu = 1 = e^{i2\pi n}$ , giving

$$\alpha = \frac{\log \mu}{\log \lambda} + \frac{i2\pi n}{\log \lambda} \quad (n = 0, 1, 2, \dots). \quad (2.4)$$

Since the renormalization group equation is linear in  $F$  a solution is given by the Fourier series

$$F(x) = x^\beta \sum_{n=0}^{\infty} k_n e^{in\omega \log x}, \quad k_n \in \mathbb{C}, \quad (2.5)$$

where  $\beta = \log \mu / \log \lambda$  and  $\omega = 2\pi / \log \lambda$ . Taking the real part of  $F$  we obtain the power law given in (2.3) for  $n = 0$ , and taking terms to  $n = 1$  gives the familiar

<sup>a</sup>The full renormalization group equation is  $F(x) = g(x) + \mu^{-1}F(\phi(x))$  (Sornette & Johansen 1997) where the nonsingular element of  $F(x)$  is characterized by some differentiable function  $g(x)$ , with  $g(0) = 0$ . We follow most authors and set  $g(x) \equiv 0$  in our analysis; for a discussion of the full equation and the significance of  $g(x)$  see Gluzman & Sornette (2002).

log-periodic power law

$$F(x) = A + |t_c - t|^\beta (B + C_1 \cos(\omega \log |t_c - t|) + C_2 \sin(\omega \log |t_c - t|)), \quad (2.6)$$

where  $A = I(t_c)$ ,  $B$ ,  $C_1$  and  $C_2$  are constants, to be fitted to the data along with  $t_c$  and  $\omega$ , the angular log-frequency of the model. The model (2.6) is often referred to as the first-order model, since it is the first of a series of models proposed originally by Sornette *et al.* (1996), and was simultaneously proposed by Feigenbaum & Freund (1996). An example of fitting the first-order model to S&P 500 index prices from the 1980s is shown in Fig. 1.

### 2.1.2. Nonlinear corrections to scaling

One of the original motivations behind the development of nonlinear corrections to scaling in the renormalization group equations stemmed from an apparent frequency shift in the first-order model when fitted to the time series of the S&P 500 index between 1980 and the crash of 1987 (Feigenbaum & Freund 1996). This is illustrated in Fig. 1 where we show the Feigenbaum and Freund fits of (2.6) starting from 1980 and from 1986 up until three weeks before the 1987 crash. By fixing  $t_c$  to be the date of the crash, they found values of the angular log-frequency  $\omega$  of 12.94 and 8.06 respectively. It should be noted that these fits have been made using absolute index price rather than the more usual use of log prices. Moreover, one can see that the fit to the time series from the beginning of 1980 seems to over-oscillate in the period between 1986 and the crash in late 1987.

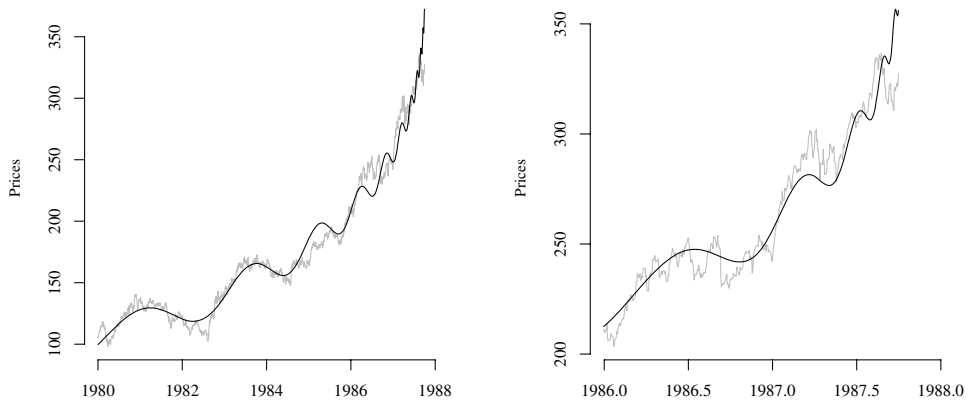


Fig. 1. First-order model fits of the S&P 500 for the periods from 1980 and 1986 up until three weeks before the crash in October 1997, with parameters taken from Feigenbaum & Freund (1996). For both plots,  $t_c = 1987.8$  (the date of the crash). The plot from 1980 (left) has parameters  $\beta = 0.20$ ,  $\omega = 12.94$ , and the plot from 1986 (right) has parameters  $\beta = 0.20$ ,  $\omega = 8.06$ . In the latter example, the authors chose to fit their model to prices rather than the more usual choice of log-price. Notice that the angular log-frequency of the shorter plot is less than that of the longer plot, and that at this frequency this plot over-oscillates closer to  $t_c$ .

To account for this apparent frequency shift, Sornette & Johansen (1997) developed a nonlinear correction to the first-order model motivated by Landau expansions from the theory of phase transitions in statistical mechanics. Taking the derivative of the power law  $F(x) = Ax^\alpha$  with respect to  $\log x$  gives  $dF(x)/d\log x = \alpha F(x)$ . Clearly, for real  $\alpha > 0$ ,  $|F(x)|$  increases with  $\log x$ , and conversely when  $\alpha < 0$ ,  $|F(x)|$  decreases with  $\log x$ . Therefore,  $F(x) = 0$  is an unstable equilibrium when  $\alpha$  is positive, and a stable equilibrium when  $\alpha$  is negative. Therefore, regarding  $\alpha$  as a bifurcation parameter, the point  $\alpha = 0$  is a bifurcation point for a saddle-node bifurcation. Complexifying, writing  $\alpha = \beta + i\omega$ , with real parameters  $\beta$  and  $\omega > 0$ , and taking an unfolding of the bifurcation to order  $O(|F|^3)$ , leads to the following model equation:

$$\frac{dF(x)}{d\ln x} = (\beta + i\omega)F(x) + (\eta + i\kappa)|F(x)|^2 F(x) + O(F(x)^5). \quad (2.7)$$

Writing  $F(x) = R(x)\exp(i\Psi(x))$ , and with a little effort (Sornette & Johansen 1997), one can solve (2.7) and recast the solution in a form analogous to the first-order equation (2.6). We refer to the resulting model as the second-order LPPL model:

$$I(t) = A + \frac{|t_c - t|^\beta}{\sqrt{1 + \left(\frac{|t_c - t|}{\Delta_t}\right)^{2\beta}}} (B + C_1 \cos \theta(t) + C_2 \sin \theta(t)),$$

where

$$\theta(t) = \omega \log |t_c - t| + \frac{\Delta_\omega}{2\beta} \log \left( 1 + \left( \frac{|t_c - t|}{\Delta_t} \right)^{2\beta} \right), \quad (2.8)$$

and the parameters in (2.7) are absorbed by the parameters in (2.8), which include the new model parameters  $\Delta_t$  and  $\Delta_\omega$ .

By taking higher order unfoldings of the bifurcation, one can obtain a hierarchy of models. For example, an unfolding up to order  $O(|F|^5)$  leads to the LPPL third-order model (Johansen & Sornette 1999a), although as the order increases so does the complexity of solving the differential equation and of fitting the resulting model. In Table 1, we show the four models in the LPPL hierarchy.

Table 1. Summary of log-periodic power law models.

| Model  | Amplitude $R(x)$  | Phase $\theta(x)$  |
|--------|---|--|
| Power  | $x^\beta$   |  |
| First  | $x^\beta$   | $\omega \log x$  |
| Second | $\frac{x^\beta}{(1 + (\frac{x}{\Delta_t})^{2\beta})^{1/2}}$                                 | $\omega \log x + \frac{\Delta_\omega}{2\beta} \log(1 + (\frac{x}{\Delta_t})^{2\beta})$   |
| Third  | $\frac{x^\beta}{(1 + (\frac{x}{\Delta_t})^{2\beta} + (\frac{x}{\Delta_t})^{4\beta})^{1/2}}$ | $\omega \log x + \frac{\Delta_\omega}{2\beta} \log(1 + (\frac{x}{\Delta_t})^{2\beta}) + \frac{\Delta'_\omega}{4\beta} \log(1 + (\frac{x}{\Delta_t})^{4\beta})$ |

Note:  $x = |t_c - t|$ .

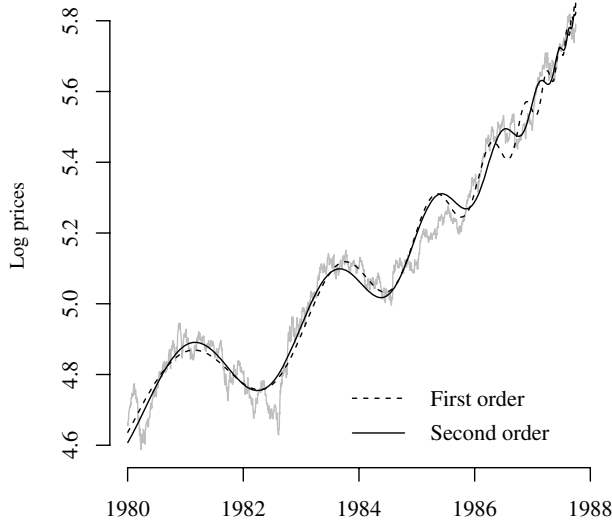


Fig. 2. First- and second-order model fits from 1980 to the 1987 S&P 500 crash, fitted using the Levenberg–Marquardt algorithm for nonlinear least squares using randomly generated parameter seed values. For a fixed value of the critical time  $t_c = 1987.80$ , being the date of the actual crash, the parameters that generate the smallest r.m.s. error in the second-order model are  $\beta = 0.68$ ,  $\omega = 8.79$ ,  $\Delta_t = 11.74$  and  $\Delta_\omega = 4.68$ . This is compared to the first-order model, which has been fitted here using the same method, with parameters  $\beta = 0.53$  and  $\omega = 12.56$ . One can clearly see how the second-order model corrects the first-order model’s over-oscillations between 1986 and the crash in late 1987.

Referring to the second-order model (2.8), one can see that as  $t$  approaches the critical time the angular log-frequency shifts from  $\omega + \Delta_\omega \rightarrow \omega$ , and that the timing of the frequency cross-over is controlled by  $\Delta_t$ . Fitting the second-order model parameters using methods described later in Sec. 3.1, we can see in Fig. 2 that the model appears to fit the oscillations quite reasonably for a fixed value of  $t_c$  when compared to a similar fit of the first-order model. It should be noted that, in contrast to Fig. 1, log prices are used here to filter out long-term exponential growth.

## 2.2. The logistic model

### 2.2.1. Solutions to the renormalization group equations and the Schröder equation

Returning to the general case of nonlinear  $\phi(x)$ , we solve (2.1) by making a change of  $x$ -coordinate. We assume there is a solution  $\psi(x)$  around  $x = 0$  of the Schröder equation  $\psi(\phi(x)) = \lambda\psi(x)$ , satisfying  $\psi(0) = 0$ . For convenience, we further assume the normalization  $\psi'(0) = 1$ . Therefore,  $\psi(x)$  conjugates  $\phi(x)$  to its linearization  $\lambda x$  around  $x = 0$ . When  $\psi(x)$  is known, it is straightforward to reduce (2.1) to



the linear case by making the change of coordinate<sup>b</sup>  $x \rightarrow \psi(x)$ , so that writing  $F(x) = \tilde{F}(\psi(x))$  gives us

$$\tilde{F}(\psi(x)) = \mu^{-1} \tilde{F}(\lambda \psi(x)), \quad \psi(x) > 0. \quad (2.9)$$

Then, from (2.2) we obtain the general solution for  $F(x)$  of (2.1), at least for  $x > 0$  sufficiently small, in the following form:

$$F(x) = \tilde{F}(\psi(x)) = \psi(x)^\beta G(\omega \log \psi(x)), \quad x > 0, \quad (2.10)$$

where  $\beta = \log \mu / \log \lambda$  and  $\omega = 2\pi / \log \lambda$ . Note that (2.10) reduces to (2.2) in the case  $\phi(x) = \lambda x$  (for which  $\psi(x) = x$ ).

In general the Schröder equation cannot be solved in closed form, but when  $\phi(x)$  is the time-1 map of an autonomous differential equation the conjugating function  $\psi(x)$  may be readily found. Consider the differential equation

$$\frac{dx}{dt} = \rho h(x), \quad x(0) = x_0, \quad h(x) = x + O(x^2). \quad (2.11)$$

The solution  $x(t)$  of (2.11) is obtained implicitly by separation of variables in terms of the function

$$H(x(t)) \stackrel{\text{def}}{=} \int \frac{dx}{h(x)} = \rho t + C, \quad (2.12)$$

where  $C$  is a constant of integration, which gives  $H(x(t)) - H(x_0) = \rho t$  when  $x(0) = x_0$ . Exponentiating this expression gives

$$\exp[H(x(t))] = \exp[H(x_0)] e^{\rho t}. \quad (2.13)$$

Now, defining  $\phi(x_0) = x(1)$ ,  $\psi(x_0) = \exp[H(x_0)]$  and  $\lambda = e^\rho$ , we obtain the functional equation  $\psi(\phi(x_0)) = \lambda \psi(x_0)$ , as required. Furthermore,  $H(x) = \log x + C + o(1)$  as  $x \rightarrow 0+$ , where  $C$  is an arbitrary constant of integration. Setting  $C = 0$ , gives, on taking the limit  $x \rightarrow 0+$ , the normalizations  $\psi(0) = 0$  and  $\psi'(0) = 1$ .

### 2.2.2. Logistic differential equation

Consider the special case of the logistic differential equation. Integrating

$$\frac{dx}{dt} = \rho x(1 + \nu x), \quad x(0) = x_0, \quad \rho = \log \lambda, \quad \lambda > 1, \quad (2.14)$$

gives the solution  $x(t) = e^{\rho t} x_0 / (1 + \nu x_0 (1 - e^{\rho t}))$  and fractional-linear time-1 map

$$\phi(x_0) = \frac{\lambda x_0}{1 + \nu x_0 (1 - \lambda)}, \quad (2.15)$$

<sup>b</sup>D. Sornette has kindly pointed out that this change of coordinate could be interpreted as a nonlinear map from calendar time to an “investor time” in line with the concept of subordination. See Geman & Ané (1996) and Mandelbrot *et al.* (1997).

which, on replacing  $x_0$  by  $x$ , gives a nonlinear perturbation of  $\phi(x) = \lambda x$ , parametrized by  $\nu$ . The conjugating function  $\psi(x)$  may also be calculated explicitly:

$$\psi(x) = \frac{x}{1 + \nu x}. \quad (2.16)$$

The behavior of the solution is dependent on the sign of  $\nu$ . There is a finite-time singularity in the solution of (2.14) when  $\nu > 0$ , and the map  $\phi(x)$  has a singularity at  $x = (\nu(\lambda - 1))^{-1}$ , but the function  $\psi(x)$  is well defined for all  $x \geq 0$ , and saturates at  $\nu^{-1}$  as  $x \rightarrow \infty$ . However, for  $\nu < 0$ ,  $\phi(x)$  is finite for  $x > 0$ , but  $\psi(x)$  has a singularity at  $x = -\nu^{-1}$ , and therefore  $F(x)$  has a singularity for  $x > 0$  as well as at  $x = 0$ .

By writing  $\nu = 1/\Delta_t$ , this nonlinear perturbation, which we call the *logistic model*, may be cast in a form comparable to those in Table 1:

$$I(t) = A + \frac{|t_c - t|^\beta}{\left(1 + \frac{|t_c - t|}{\Delta_t}\right)^\beta} [B + C_1 \cos \theta(t) + C_2 \sin \theta(t)], \quad (2.17)$$

$$\theta(t) = \omega \log |t_c - t| - \omega \log \left(1 + \frac{|t_c - t|}{\Delta_t}\right). \quad (2.18)$$

Whilst the second-order and logistic models coincide with the first-order model for small  $|t_c - t|$ , the properties of the logistic model (2.17) for large  $|t_c - t|$  are distinct from those of (2.8). For  $\Delta_t > 0$ , both models lead to saturation of the amplitude with limit  $\Delta_t^\beta$ . However, model (2.8) exhibits a frequency shift  $\omega \rightarrow \omega + \Delta_\omega$  in the log-periodic oscillations (since  $\theta(t) \sim (\omega + \Delta_\omega) \log |t_c - t| - \Delta_\omega \log \Delta_t$  for large  $|t_c - t|$ ), whilst in model (2.17)  $\theta(t) \rightarrow \omega \log \Delta_t$  as  $|t_c - t| \rightarrow \infty$  and the log-periodic oscillations die away. This difference can be used to determine which model is of more practical usefulness in analyzing various asset bubbles and anti-bubbles. For  $\Delta_t < 0$ , the logistic model has a secondary critical point at  $|t_c - t| = |\Delta_t|$ .

This means that the log-periodic oscillations leading up to this secondary critical point grow without bound. Throughout the literature on this topic, the aim has always been to consider cases only where  $F(x)$  remains bounded, since singular behavior is not seen in real world observations. This suggests we should neglect cases where  $\Delta_t < 0$ . However, as we will see later, allowing this situation can lead to some interesting results, and the model can nevertheless be used to obtain remarkable fits in some cases, implying both beginning and end points to regions of cooperative behavior. It should be noted that the LPPL and logistic models are applicable only during these periods of cooperative behavior and are not valid outside the bounds of  $t > t_c$  for a bubble and  $t < t_c$  for an anti-bubble. Moreover, for  $\Delta_t < 0$ , the logistic model is additionally not valid outside the bounds  $t > t_c + \Delta_t$  and  $t < t_c + |\Delta_t|$  for bubble and anti-bubble respectively. Applications of the logistic model are described in Sec. 3.

### 2.2.3. Perturbation of the logistic differential equation

By taking higher-order nonlinearities in the differential equation (2.14), it is possible to build a hierarchy of models as in the LPPL approach. For example, taking a perturbation of the logistic differential equation

$$\frac{dx}{dt} = \rho x(1 + \nu x)(1 + \sigma x), \quad |\sigma| < |\nu|, \quad (2.19)$$

gives (2.10) with

$$\psi(x) = \frac{x(1 + \sigma x)^{\frac{\sigma}{\nu - \sigma}}}{(1 + \nu x)^{\frac{\nu}{\nu - \sigma}}}. \quad (2.20)$$

Writing  $\nu = 1/\Delta_t$  and  $\sigma = \epsilon/\Delta_t$ , gives

$$\psi(x) = \frac{x \left(1 + \frac{\epsilon x}{\Delta_t}\right)^{\frac{\epsilon}{1 - \epsilon}}}{\left(1 + \frac{x}{\Delta_t}\right)^{\frac{1}{1 - \epsilon}}}, \quad (2.21)$$

and so we have a *modified logistic model*:

$$I(t) = A + \left( \frac{|t_c - t| \left(1 + \frac{\epsilon |t_c - t|}{\Delta_t}\right)^{\epsilon/(1 - \epsilon)}}{\left(1 + \frac{|t_c - t|}{\Delta_t}\right)^{1/(1 - \epsilon)}} \right)^{\beta} [B + C_1 \cos \theta(t) + C_2 \sin \theta(t)], \quad (2.22)$$

$$\theta(t) = \omega \left( \log |t_c - t| - \frac{1}{1 - \epsilon} \left(1 + \frac{|t_c - t|}{\Delta_t}\right) + \frac{\epsilon}{1 - \epsilon} \left(1 + \frac{\epsilon |t_c - t|}{\Delta_t}\right) \right). \quad (2.23)$$

Although not necessary for the purposes of this paper, it is straightforward to adapt the techniques described in this paper to fit the modified logistic model (and other models obtained from other perturbations of the logistic differential equation) to financial time series displaying the characteristics of bubbles and anti-bubbles.

## 3. Comparison of Model Fitting for S&P 500 and Nikkei 225 Indices

### 3.1. Example 1: S&P 500 bubble from 1980 to October 1987

Our first case study is the long bull market in the 1980s leading to the crash of October 1987. The S&P 500 index during this period has been used extensively as a testing ground for models of financial crashes.

#### 3.1.1. Fitting the models

Fitting these nonlinear models to the observed data is a difficult computational task, and it is unclear by what measure a set of parameter choices could be described as providing a “best fit”. Bree *et al.* (2013) give a very interesting account of the

challenges faced, and take a close look at fitting LPPLs to financial time series using the Levenberg–Marquardt algorithm (Gavin 2011). It is clear that very small changes in some parameters have a very large effect on the r.m.s errors between the model and the data points, and on the other hand, for some parameters, very large changes barely affect the r.m.s errors at all. Therefore, one can find many widely varying parameter sets that have essentially equivalent r.m.s errors.

In the formulae for the various models in Table 1, (2.17) and (2.22), the index  $I(t)$  is a linear function of several of the parameters, e.g.  $A$ ,  $B$ ,  $C_1$  and  $C_2$ . In our work below, we refer to these parameters as the *linear parameters* of the model. We call the other parameters, most particularly  $t_c$ ,  $\omega$  and  $\beta$ , *nonlinear parameters*. In common with the approach of Sornette and co-workers, for each choice of nonlinear parameters, we use a standard least-squares algorithm to fit the remaining linear parameters, thereby reducing significantly the computational task involved.

In keeping with a large portion of practitioners in this field, we have used the Levenberg–Marquardt nonlinear least-squares algorithm to create the plots in Fig. 2. Bree *et al.* (2013) found that varying  $t_c$  had only a small effect on the r.m.s. errors, but  $t_c$  was very sensitive to the inclusion or exclusion of various data points. Therefore, following Feigenbaum & Freund (1996), we fixed the value of the nonlinear parameter  $t_c = 1987.80$ . We then generated many thousands of parameter values for the remaining nonlinear parameters  $\omega$  and  $\beta$  in the case of the first-order LPPL model, along with  $\Delta_t$  and  $\Delta_\omega$  in the case of the second-order model. For each randomly generated set of nonlinear parameters we found the corresponding linear parameters analytically by functional linear least squares. The Levenberg–Marquardt algorithm then was applied to each of these parameter sets and the set that minimized the r.m.s. errors for each model was chosen. The results shown in Fig. 2 illustrate the advantage of the second-order LPPL model to the first-order model in the specific example of the S&P 500 index time series between 1980 and the crash in October 1987.

When comparing the hierarchy of LPPL models with the logistic model, we avoided the computational difficulties of the Levenberg–Marquardt algorithm, and focused on the interdependencies of a subset of nonlinear parameters. As in several works by Sornette and his co-workers (Sornette 2003) and Vandewalle *et al.* (1998b), our approach uses the gross features of the time series to obtain a relationship between the critical time  $t_c$  and the angular log-frequency  $\omega$ .

Denoting by  $t_k$  the times at which the time series is a local maximum (as determined by eye and ignoring smaller scale oscillations), and assuming a constant angular log-frequency in the log-periodic oscillations, we may use two pairs of consecutive points,  $(t_1, t_2)$  and  $(t_3, t_4)$ , to estimate the critical time using the cross-ratio condition  $(t_c - t_1)/(t_c - t_2) = (t_c - t_3)/(t_c - t_4)$ . This gives an estimate for the critical time  $t_F$  implied by the first-order model

$$t_F = \frac{c_2}{c_1}, \quad c_1 = t_1 + t_4 - t_2 - t_3, \quad c_2 = t_1 t_4 - t_2 t_3, \quad (3.1)$$

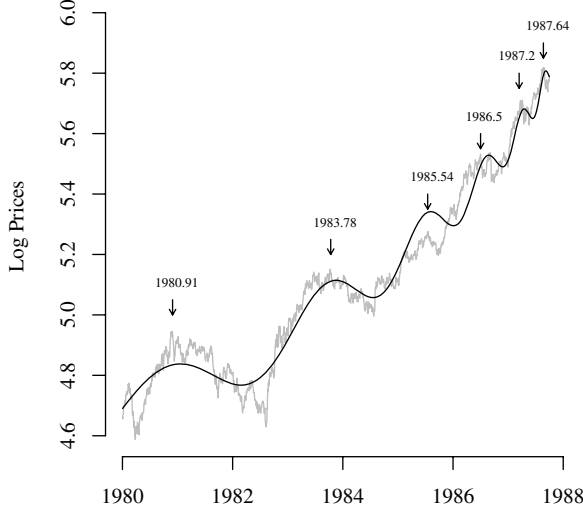


Fig. 3. First-order fit of the S&P 500 index 1980–1988. The times corresponding to peaks in the oscillations are marked by arrows. Assuming a constant angular log-frequency  $\omega$  throughout the period, and requiring the model to fit the log-frequency of the peaks between 1980 and 1984 and between 1986 and 1987.5, we derive the value  $t_F = 1988.30$  where  $t_1 = 1980.91$ ,  $t_2 = 1983.78$ ,  $t_3 = 1986.5$ , and  $t_4 = 1987.2$  in (3.1). From this value of  $t_F$ , we find  $\omega = 12.74$ , and finally  $\beta = 0.4$  is used to minimize the r.m.s. errors.

and it follows that an estimate for first-order angular log-frequency,  $\omega_F$ , is given by

$$\omega_F = \frac{2\pi}{\log(t_F - t_1) - \log(t_F - t_2)}. \quad (3.2)$$

This method is illustrated for the S&P 500 index in the period 1980–1988 in Fig. 3. If we require the first-order model to fit the angular log-frequencies in the period between 1980 and 1984 and between 1986 and 1987.5, thus “solving” the issue of the apparent frequency shift shown in Fig. 1, then, by (3.1), the consecutive peaks at 1980.91 and 1983.78, and later at 1986.5 and 1987.2, give a unique value for  $t_F = 1988.30$ . Having found this value of  $t_F$ , from (3.2) we find that  $\omega = 12.74$ , being very close to the value found in Feigenbaum & Freund (1996) for the plot between 1980 and 1987.8. However, the value of  $t_F$  does not seem to be a good predictor of the time of the crash (1987.80). Here, we are not claiming that the critical time should indeed be the actual time of the crash; we are merely confirming that the first-order model cannot fit the gross features of both the long-term and the short-term time series with a constant angular log-frequency if one assumes the time of the crash and the critical time are coincident.

### 3.1.2. Frequency shifting in S&P 500 and the logistic model

As mentioned above, the second-order LPPL model was developed to explain the apparent frequency shift in the log-periodic oscillations of the S&P 500 index

observed by Feigenbaum & Freund. However, we note that by moving the critical time out further into 1988, the first-order model can fit the long-term data very well. Nevertheless, if it is required that the fitted critical time  $t_c$  is close to the actual time of the crash, the logistic model's nonlinear perturbation of the first-order LPPL model may also be of some assistance in this particular case.

In fitting the logistic model, a similar method can be used to obtain an estimate for  $t_L$ , the critical time associated with the logistic model. Using the times  $t_1, \dots, t_4$ , as above, and the relation for two consecutive times  $t_k$  and  $t_{k+1}$

$$\omega_L = \frac{2\pi}{\log\left(\frac{t_L - t_k}{1 + \frac{t_L - t_k}{\Delta_t}}\right) - \log\left(\frac{t_L - t_{k+1}}{1 + \frac{t_L - t_{k+1}}{\Delta_t}}\right)}, \quad (3.3)$$

gives, on exponentiating,

$$\left(\frac{t_L - t_1}{t_L - t_2}\right) \left(\frac{\Delta_t + t_L - t_2}{\Delta_t + t_L - t_1}\right) = \left(\frac{t_L - t_3}{t_L - t_4}\right) \left(\frac{\Delta_t + t_L - t_4}{\Delta_t + t_L - t_3}\right), \quad (3.4)$$

which, on expanding gives the quadratic equation in  $t_L$ :

$$-c_1 \Delta_t t_L^2 + (c_1 \Delta_t^2 + 2c_2 \Delta_t) t_L - c_2 \Delta_t^2 + c_3 \Delta_t = 0, \quad (3.5)$$

where  $c_1, c_2$  are as above and  $c_3 = t_1 t_2 t_3 - t_1 t_2 t_4 - t_1 t_3 t_4 + t_2 t_3 t_4$ . With a little rearranging, we find a relationship in terms of the control parameter,  $\Delta_t$  and the first-order estimated critical time,  $t_F$

$$t_L = t_F - \frac{\Delta_t}{2} \pm \sqrt{\frac{\Delta_t^2}{4} + \kappa}, \quad (3.6)$$

where  $\kappa = (c_2^2 + c_1 c_3)/c_1^2 > 0$ .

If one takes the positive root, then, as  $\Delta_t \rightarrow \infty$ , the critical time  $t_L \rightarrow t_F$  from above, and, as  $\Delta_t \rightarrow -\infty$ ,  $t_L \rightarrow \infty$ . This means that, by taking this positive root, it is impossible to find a value for  $t_L < t_F$ . On the other hand, if one takes the negative root, then, as  $\Delta_t \rightarrow \infty$ , the critical time  $t_L \rightarrow -\infty$ , but as  $\Delta_t \rightarrow -\infty$ ,  $t_L \rightarrow t_F$  from below. Since by taking the positive root, one cannot find a critical time closer to the time of the actual crash than  $t_F$ , and, if it is important that  $t_L$  is close to the actual crash time, we must take the negative root.

Equation (3.6) gives a relation between  $t_L$  and  $\Delta_t$ . In the case of the S&P index between 1980 and 1987.8 we further assume  $t_L = 1987.8$ , the time of the crash. This enables us to derive  $\Delta_t = -15.70$  from (3.6) with  $t_1 = 1980.91, t_2 = 1983.78, t_3 = 1986.5$ , and  $t_4 = 1987.2$ . Then from (3.3) we have  $\omega = 7.65$ .

This fit is shown in Fig. 4. Although to the eye the fit is not as good as the fit that can be achieved with the first-order model with  $t_F = 1988.30$ , the logistic model with  $t_L = 1987.80$  does not show the over-oscillation of the first-order model with  $t_F = 1987.80$ . However, the logistic model does not perform as well as the second-order LPPL model of Sornette & Johansen (1997). Nevertheless, if one relaxes the

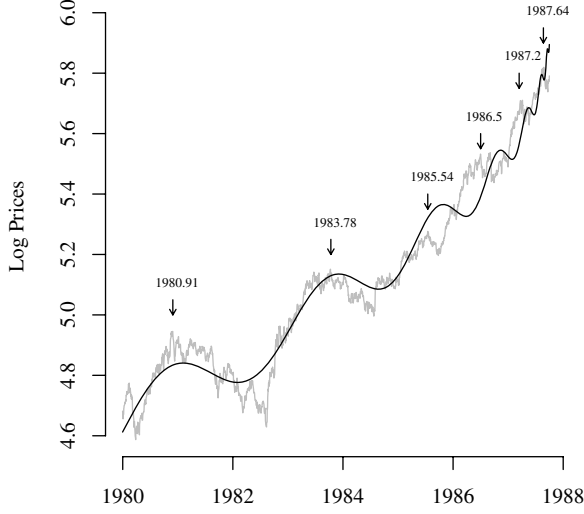


Fig. 4. Fit of the logistic model to S&P 500 1980–1988, with peaks marked by arrows. The parameter  $\Delta_t$  is calculated from (3.5) the logistic model assuming a constant angular log-frequency  $\omega$  and  $t_c = 1987.8$ , which gives  $\Delta_t = -15.7$ , and  $\omega = 7.65$ . The logistic model picks out the main features more accurately than the first-order model when one fixes the critical time to that of the crash itself, but not as well as the second-order model with the additional parameter,  $\Delta_\omega$ .

requirement that the critical time is close to the observed crash, then the S&P 500 series with the apparent frequency shifting observed by Feigenbaum & Freund can be well described by a constant angular log-frequency with the nonlinear perturbation given by the logistic model. We observe, in fact, that the logistic model gives an equally good account of the observed data as the second-order model if one allows the value of  $t_c$  to vary away from the actual crash date. To match the second-order model in terms of r.m.s. errors, the value of  $t_c$  for the logistic model is approximately 1988.

### 3.2. *Example 2: Nikkei 225 anti-bubble from 1990 to 2003*

Our second case study is the period 1990–2003 in which the Japanese stock market experienced a long bear run following the Japanese asset price bubble of the late 1980s. In the following analysis, we consider time-series data between the known start and end dates of the anti-bubble.

For anti-bubbles, we expect the critical time,  $t_c$  to be located prior to the formation of the herding phenomenon which leads to a faster than exponential decline in asset prices. Therefore, each of the equations describing the LPPL hierarchy or logistic models must be rewritten to such that  $t_c - t$  is replaced with  $t - t_c$ . Rewriting (3.4) as

$$\left(\frac{t_2 - t_L}{t_1 - t_L}\right) \left(\frac{\Delta_t + t_1 - t_L}{\Delta_t + t_2 - t_L}\right) = \left(\frac{t_4 - t_L}{t_3 - t_L}\right) \left(\frac{\Delta_t + t_3 - t_L}{\Delta_t + t_4 - t_L}\right), \quad (3.7)$$

we obtain the relationship between  $t_L$  and  $\Delta_t$  for the logistic model of an anti-bubble regime:

$$\Delta_t = \frac{c_1 t_L^2 - 2c_2 t_L - c_3}{c_1 t_L - c_2}, \quad (3.8)$$

where, as before,

$$c_1 = t_1 + t_4 - t_2 - t_3, \quad (3.9)$$

$$c_2 = t_1 t_4 - t_2 t_3, \quad (3.10)$$

$$c_3 = t_1 t_2 t_3 - t_1 t_2 t_4 - t_1 t_3 t_4 + t_2 t_3 t_4, \quad (3.11)$$

and the angular log-frequency is given by

$$\omega = \frac{2\pi}{\log \left( \frac{t_{k+1} - t_L}{1 + \frac{t_{k+1} - t_L}{\Delta_t}} \right) - \log \left( \frac{t_k - t_L}{1 + \frac{t_k - t_L}{\Delta_t}} \right)}. \quad (3.12)$$

As with the previous example, we have picked out the gross features of the Nikkei 225 index between the origin of the anti-bubble at the beginning of 1990s to the end of the long down-turn in 2003. However, in this example the peaks seem to be less well defined than that of the S&P 500 index between 1980 and 1988. Therefore, we have marked the periodicity using the troughs, which are better defined.

We can determine a value for  $\Delta_t$  using (3.8) by taking two pairs of consecutive troughs, and specifying the value of  $t_L$ . In an anti-bubble regime, we expect the critical time to occur before the start of the regime. In this particular time series there is a sharp peak in the observations just before the beginning of the 1990s, so, in this case, we have made the decision that, first, it is important to set the critical time at the point where the anti-bubble begins, and, second, that this beginning is at  $t_L = 1989.95$ .

As can be seen from Fig. 5, there are many pairs of consecutive troughs that can be chosen, and in contrast to the S&P 500 example where an attempt was made to correct the over-oscillations of the first-order model in a particular period, in the Nikkei 225 problem there are no *a priori* reasons to choose one pair over any other.

Therefore, we have found values of  $\Delta_t$  and the corresponding values for  $\omega$  using (3.8), and all combinations of consecutive pairs of troughs. The results of these calculations can be seen in Table 2. Apart from a few outlying data points (corresponding to pairs of troughs at the beginning and the end of the time period), the values of  $\Delta_t$  calculated from picking out these troughs by eye show remarkable consistency over a very long time span. From this data, we are able to derive a limited distribution of values for  $\Delta_t$  and  $\omega$ , by including only the data from Table 2 corresponding to times within the data set being analyzed.

In an anti-bubble regime, if the value of  $t_c$  signals the beginning of the anti-bubble, then the singularity at  $t = t_c + |\Delta_t|$  signals its end. Therefore, we must also



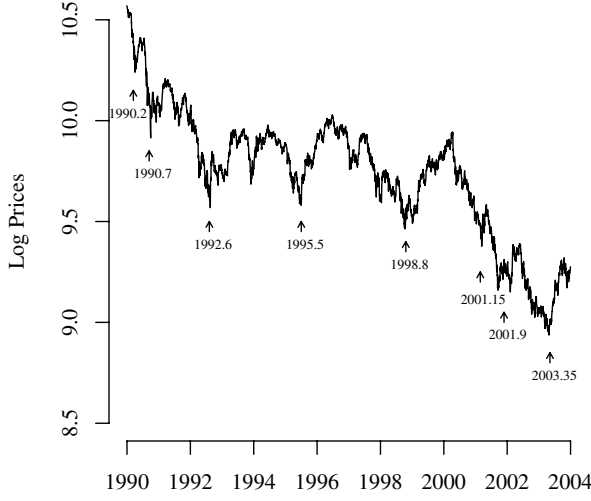


Fig. 5. The peaks of the Nikkei 225 index are not very well defined so, in this case, we use the troughs as markers for the periodicity. As one can see, we have included a marker at  $t = 2001.15$  but neglected to do so for the intermediate trough around  $t = 1994$ . The reason for this is that the latter dip seems not be in keeping with a general downward trend. Clearly this is a largely subjective judgement.

Table 2. Implied values for  $\omega$  and  $\Delta_t$  from pairs of consecutive troughs.

| $t_1$   | $t_2$   | $t_3$   | $t_4$   | $\omega$ | $\Delta_t$ |
|---------|---------|---------|---------|----------|------------|
| 1990.20 | 1990.70 | 1990.70 | 1992.60 | 6.12     | 6.38       |
| 1990.20 | 1990.70 | 1992.60 | 1995.50 | 5.49     | -11.35     |
| 1990.20 | 1990.70 | 1995.50 | 1998.80 | 5.51     | -12.28     |
| 1990.20 | 1990.70 | 1998.80 | 2001.15 | 5.52     | -12.80     |
| 1990.20 | 1990.70 | 2001.15 | 2001.90 | 5.51     | -12.33     |
| 1990.20 | 1990.70 | 2001.90 | 2003.35 | 5.54     | -14.21     |
| 1990.70 | 1992.60 | 1992.60 | 1995.50 | 3.98     | -7.76      |
| 1990.70 | 1992.60 | 1995.50 | 1998.80 | 4.27     | -10.75     |
| 1990.70 | 1992.60 | 1998.80 | 2001.15 | 4.35     | -12.20     |
| 1990.70 | 1992.60 | 2001.15 | 2001.90 | 4.35     | -12.20     |
| 1990.70 | 1992.60 | 2001.90 | 2003.35 | 4.43     | -13.94     |
| 1992.60 | 1995.50 | 1995.50 | 1998.80 | 5.84     | -12.78     |
| 1992.60 | 1995.50 | 1998.80 | 2001.15 | 5.89     | -13.01     |
| 1992.60 | 1995.50 | 2001.15 | 2001.90 | 5.74     | -12.37     |
| 1992.60 | 1995.50 | 2001.90 | 2003.35 | 6.16     | -14.37     |
| 1995.50 | 1998.80 | 1998.80 | 2001.15 | 6.03     | -13.09     |
| 1995.50 | 1998.80 | 2001.15 | 2001.90 | 5.55     | -12.34     |
| 1995.50 | 1998.80 | 2001.90 | 2003.35 | 6.81     | -14.56     |
| 1998.80 | 2001.15 | 2001.15 | 2001.90 | 4.35     | -12.20     |
| 1998.80 | 2001.15 | 2001.90 | 2003.35 | 9.07     | -15.25     |
| 2001.15 | 2001.90 | 2001.90 | 2003.35 | -13.09   | -10.16     |

exclude values of  $\Delta_t$  that imply an end to the anti-bubble prior to the end of the data set. If one accepts this argument, one must also exclude any positive value of  $\Delta_t$  which implies the end of the anti-bubble being located before its start. However, logistic anti-bubble models with  $\Delta_t > 0$  do not exhibit finite-time singularities but rather saturate as  $t \rightarrow \infty$ , which does not describe well the long-term progression of the Nikkei 225 index from 1990 to 2003.

### 3.2.1. Fitting the logistic model with Markov Chain Monte Carlo (MCMC)

In the logistic model there are four nonlinear parameters to find, namely  $t_c$ ,  $\beta$ ,  $\omega$  and  $\Delta_t$ . For this example, we have fixed the critical time  $t_c = 1989.95$  and we have used the information about the distribution of both  $\omega$  and  $\Delta_t$  derived from Table 2. We also know that  $0 < \beta < 1$  so that the model remains bounded at  $t_c$  and is singular in its first derivative at  $t_c$ , and we use Student's  $t$ -distribution for  $\omega$  and  $\Delta_t$ . Hence we choose reasonable prior distributions for these parameters

$$\beta \sim U(0, 1), \quad \omega \sim t(\bar{\omega}, s_\omega, \nu), \quad \Delta_t \sim t(\bar{\Delta}_t, s_{\Delta_t}, \nu), \quad (3.13)$$

where the means  $\bar{\omega}$  and  $\bar{\Delta}_t$ , and the sample variances  $s_{\Delta_t}^2$  and  $s_\omega^2$  are calculated directly from Table 2 with respect to the data set in question making the above exclusions, and the degrees of freedom,  $\nu$ , is the number of observations minus 1. From here, we use MCMC simulation (using the JAGS software implemented in R) to generate posterior distributions for each of the nonfixed parameters (including the linear parameters). We note that in practice one needs to truncate the distribution for  $\Delta_t$  such that a singularity cannot occur within the time span of the data series.

As an illustration, we take two examples from Johansen & Sornette (1999a), where the authors have fitted the second-order LPPL model to the period between 1990 and 1995.5 and the third-order model to the period between 1990 and 1999. We compare these fits with the results obtained from fitting the logistic model using the above method.

First, we exclude the trough pairs that are not suitable for this particular data set. As can be seen from Table 3, there are only two entries that are suitable, giving  $\bar{\Delta}_t = -9.55$  and  $s_{\Delta_t}^2 = 6.46$ , and  $\bar{\omega}_t = 4.73$  and  $s_\omega^2 = 1.13$ . Then, as can be seen from Fig. 6, using the prior distributions in (3.13), the MCMC simulation produces approximately normal posterior distributions for the parameters  $\omega$  and  $\beta$ , and a skew-normal distribution for  $\Delta_t$  as follows:

$$\beta \sim N(0.02, 0.04), \quad \omega \sim N(5.47, 0.10), \quad \Delta_t \sim SN(-35.80, 8.29, 0.50). \quad (3.14)$$

Table 3. Implied values for  $\omega$  and  $\Delta_t$  from pairs of consecutive troughs for the period between 1990 and 1995.5 excluding positive values of  $\Delta_t$ .

| $t_1$   | $t_2$   | $t_3$   | $t_4$   | $\omega$ | $\Delta_t$ |
|---------|---------|---------|---------|----------|------------|
| 1990.20 | 1990.70 | 1992.60 | 1995.50 | 5.49     | -11.35     |
| 1990.70 | 1992.60 | 1992.60 | 1995.50 | 3.98     | -7.76      |

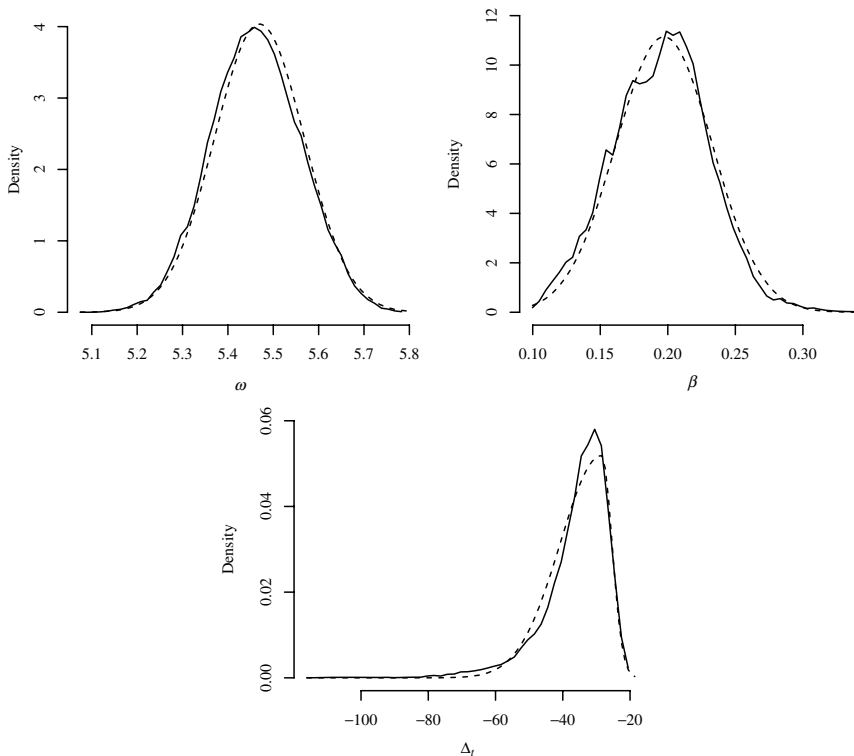


Fig. 6. Density plots of posterior distributions for logistic model fitted between 1990 and 1995.5 with approximate distributions (dotted lines), normal for  $\omega$  and  $\beta$  and skew-normal for  $\Delta_t$ .

Using the mode of each of the distributions in (3.14) and the fixed value for  $t_c$  we can find the linear parameters analytically by minimizing the r.m.s. errors. It is interesting to note that 95% confidence interval for  $\Delta_t$  is between  $-61.97$  and  $-22.65$ . Following our earlier argument, this puts the date for the end of the crash between 2012.65 and 2051.97. However, one can see that the anti-bubble ended in the first half of 2003 and the calculated value, i.e. the mode of the posterior distribution, is  $\Delta_t = -20.31$  which is not representative of the end date of the anti-bubble.

The results of this model-fitting is compared to that of the second-order LPPL model in Fig. 7. This second-order model fit is recreated from the parameters given in Johansen & Sornette (1999a), namely,

$$t_c = 1989.97, \quad \beta = 0.41, \quad \omega = 4.8, \quad \Delta_t = 9.5, \quad \Delta_\omega = 4.9. \quad (3.15)$$

One can see that the logistic model fits the 5.5 years of data very well, as does the second-order model. Figure 7 shows how the observations develop compared to the predictive progression of the fitted models out until 2002.5. Although neither of the models provide a meaningful fit to the future, nonfitted data, one can see

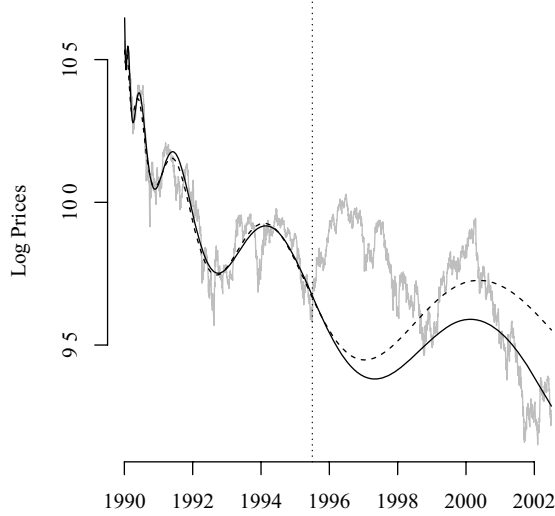


Fig. 7. Second-order model using Johansen & Sornette (1999a) parameter fit (dashed line) and logistic models (solid line) using posterior mean of each parameter from MCMC simulation as in (3.14). The data set used is between 1990 and 1995.5, the end of this period being marked by a vertical dotted line. As can be seen the second-order and logistic models fits are similar for this period. However, when the time series is extended out to 2002.5, at first sight the logistic model seems to have more accurate long-term predictive properties.

that as the second-order model saturates it cannot follow the steepening downward trend of the time series. Conversely, as  $t \rightarrow t_c - \Delta_t$ , the logistic model begins to oscillate more rapidly and tends to the unbounded downside. As we will see, this seems to more accurately match the actual development of this anti-bubble.

Second, we look at nine years of observed data from the Nikkei 225 index from 1990, and compare the results of the logistic model with those of the third-order LPPL model.

Following the same method as above, we look for consecutive pairs of troughs prior to 1999 from Table 2 that give negative values of  $\Delta_t$ . The set of pairs are shown in Table 4 and give  $\bar{\Delta}_t = -10.98$  and  $s_{\Delta_t}^2 = 3.87$ , and  $\bar{\omega}_t = 5.02$  and  $s_{\omega_t}^2 = 0.69$ . Again, using the prior distributions in (3.13), the MCMC simulation produces approximately normal posterior distributions (Fig. 8) for two of the parameters ( $\omega$  and  $\Delta_t$ ), and a skew-normal distribution for the third parameter  $\beta$ , as follows:

$$\beta \sim SN(0.013, 0.001, 2.5), \quad \omega \sim N(5.68, 0.06), \quad \Delta_t \sim N(-12.59, 0.22). \quad (3.16)$$

Again, we compare the results of this model fitting to that of a LPPL model, in this case, the third-order model. This is shown in Fig. 9. This third-order model fit is again recreated from the parameters given in Johansen & Sornette (1999a). For this nine-year plot, the authors retain the parameters  $t_c$ ,  $\beta$  and  $\omega$  from the previous 5.5 year second-order fit and vary only  $\Delta_t$ ,  $\Delta_\omega$ ,  $\Delta'_t$  and  $\Delta'_\omega$ . The curve is fitted with

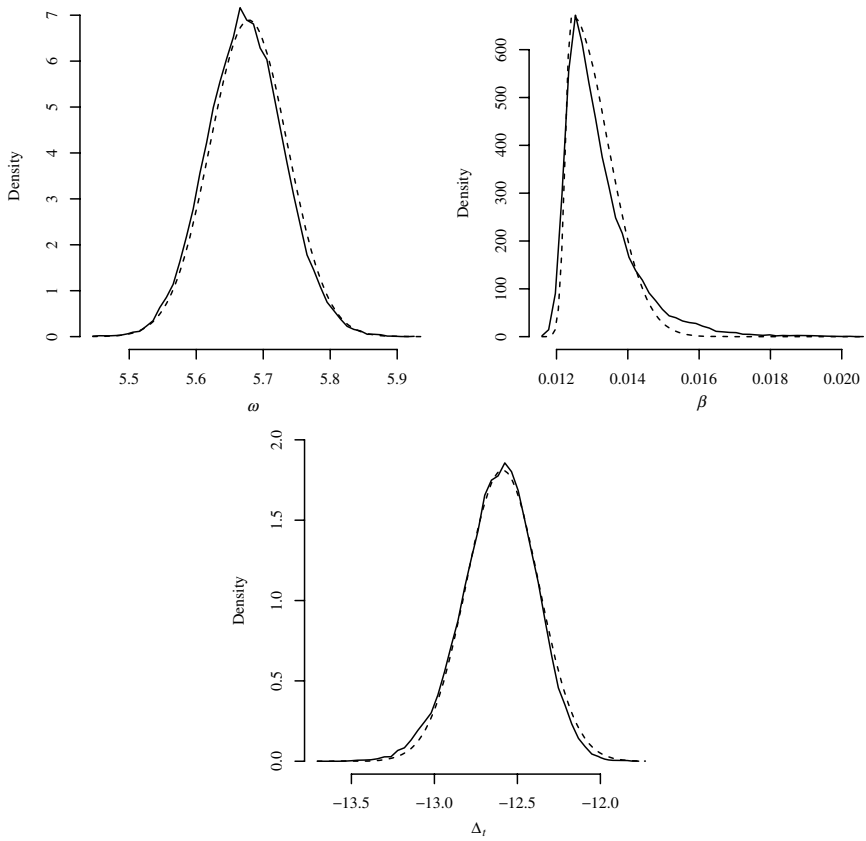


Fig. 8. Density plots of posterior distributions for logistic model fitted between 1990 and 1999 with approximate normal distributions (dotted lines) for the  $\omega$  and  $\Delta_t$  parameters. The parameter  $\beta$  is approximately skew-normally distributed.

Table 4. Implied values for  $\omega$  and  $\Delta_t$  from pairs of consecutive troughs suitable for data between 1990 and 1999 excluding positive values of  $\Delta_t$ .

| $t_1$   | $t_2$   | $t_3$   | $t_4$   | $\omega$ | $\Delta_t$ |
|---------|---------|---------|---------|----------|------------|
| 1990.20 | 1990.70 | 1992.60 | 1995.50 | 5.49     | -11.35     |
| 1990.20 | 1990.70 | 1995.50 | 1998.80 | 5.51     | -12.28     |
| 1990.70 | 1992.60 | 1992.60 | 1995.50 | 3.98     | -7.76      |
| 1990.70 | 1992.60 | 1995.50 | 1998.80 | 4.27     | -10.75     |
| 1992.60 | 1995.50 | 1995.50 | 1998.80 | 5.84     | -12.78     |

the following values:

$$\Delta_t = 4.34, \quad \Delta_\omega = -3.10, \quad \Delta'_t = 7.83, \quad \Delta'_\omega = 23.4. \quad (3.17)$$

Figure 9 shows the third-order model more accurately fitting the amplitudes of the observed data than the logistic model. However, the logistic model picks out

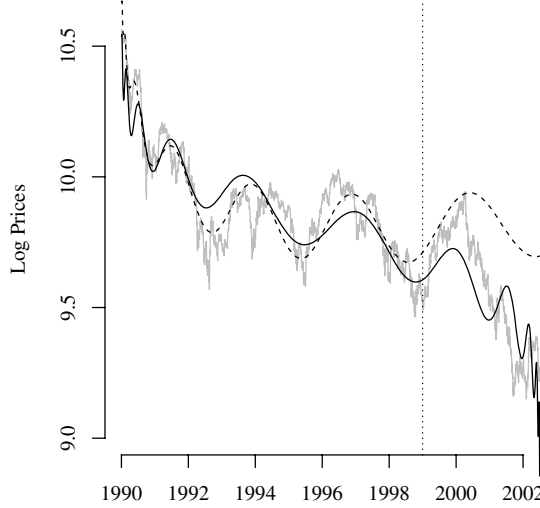


Fig. 9. Third-order model using the Johansen & Sornette (1999a) parameter fit (dashed line) and the logistic models (solid line) using posterior mean of each parameter from the MCMC simulation. The data set used is between 1990 and 1999, the end of this period being marked by a vertical dotted line. Note the remarkable predictive quality of the logistic model when the data series is extended to future, nonfitted data between 1999 and 2002.5. Having a singularity at  $t_c + |\Delta_t|$  (for  $\Delta_t < 0$ ), the logistic model does not apply past the end of the anti-bubble.

the periodicity of the observed data equally as well as the third-order model, and it manages to do this with three fewer nonlinear parameters. With both the second- and third-order models, the fits to the data are generally good because as  $t$  moves away from  $t_c$  their angular log-frequencies are constantly changing. Conversely, the logistic model is able to track the periodicity by holding its log-frequency constant but assuming a second singularity at a distance  $|\Delta_t|$  from  $t_c$ .

The striking feature of the logistic model is how the progression of the model closely follows the development of the nonfitted future data over the subsequent 3.5 years. By this time the third-order model is beginning to saturate and offers no predictive value past the beginning of 2000. However, the logistic model's increasingly rapid oscillations and steep descent describe the path of this index very well to the naked eye. Of course, this is an out of sample fit rather than a prediction, and it should be noted that in May 1999 the third-order LPPL model produced a very accurate *ex ante* prediction of the trend reversal in the Nikkei index over the subsequent year (Johansen & Sornette 2000).

Furthermore the 95% confidence limit for the distribution of  $\Delta_t$  is between  $-12.18$  and  $-13.05$  implying an end to the anti-bubble between 2002.18 and 2003.05. As it happened, the anti-bubble came to an end in the first half of 2003, around 2003.3.

Finally, we show in Fig. 10 a fit of the logistic model using the method employed for the previous two examples over the whole time period 1990–2003. Given the

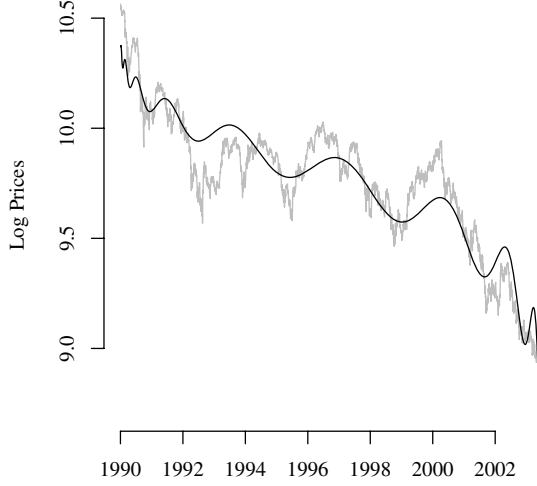


Fig. 10. The logistic model is fitted from 1990 to the beginning of 2003. Although the amplitude of the oscillations is not accurately described by the model the periodicity seems to be very good. Additionally, the found value of  $\Delta_t$  implies an end to the anti-bubble at 2003.66. As stated the Nikkei 225 began its upswing shortly before this around 2003.3.

saturation features of the second- and third-order LPPL equations, one would not expect these models to perform well, but, intriguingly, the time series seems to suit the logistic model extraordinarily well.

#### 4. Summary and Suggestions for Further Work

We have shown how the suite of log-periodic power-law models for asset bubbles and anti-bubbles can be extended to include models derived from solutions to the Schrödinger equation which satisfy the original renormalization group equations. In particular, we have derived models from the logistic differential equation, which we have called the logistic model and modified logistic model. We do not claim that these models describe reality better (or worse) than the established models, but we believe them to be important additions to the modeling toolbox.

Indeed, on examining the S&P 500 index bubble between 1980 and the crash of October 1987, we have observed that the logistic model may have some advantages in terms of flexibility over the first-order LPPL model in situations where there is a required value for the critical time, and where, on first sight, the observed data appears to have oscillations with varying log-frequencies. Additionally, the logistic model has the advantage of fewer parameters over the second- and third-order extensions.

In the case of the Nikkei 225 index anti-bubble between 1990 and 2003, the addition of a second singularity in the logistic model derived by taking negative values of  $\Delta_t$  provides a model with features that are remarkably predictive over long time-periods for this particular data set.

Generally, the use of solutions of the renormalization group formalism corresponding to a nonlinear  $\phi(x)$  is appealing, as is the implied beginning to the bubble (and end to the anti-bubble) given by  $\Delta_t < 0$ . However, in this paper we have not gone far in investigating whether or not these ideas have any concrete advantages over any other method of modeling bubble or anti-bubble regimes. In future work, one could take a more rigorous approach to the determination of the logistic model's parameters over a wide range of faster-than-exponential growth/decline regimes, and discover whether or not there is any value in extending these models with further perturbations as we have done in our derivation of the modified logistic model. A key test of the new models will be whether they can successfully make *ex ante* predictions of the end of an asset bubble and/or anti-bubble.

Another promising research direction is to use MCMC or other statistical methods to fit LPPL change-point models to both the pre- and post-crash time-series to understand better the onset of asset bubbles, the bubble-anti-bubble transition and the resumption of 'business as usual' after the end of the anti-bubble.

Finally, another interesting feature of the logistic model fit to the S&P 500 index between 1980 and the October 1987 crash is that the fitted value of  $\Delta_t = -15.7$  implies a singularity  $I(t) \rightarrow -\infty$  when  $t \rightarrow 1972.1$ . Clearly this feature is not seen in the observed data, and by plotting the model further back in the time series, the logistic model provides no particular insight into the pre-1980s development of the bubble, as we have seen in the predictive aspects of the logistic model when fitted to the Nikkei 225 index anti-bubble between 1990 and 1999. This suggests that anti-bubbles cannot simply be regarded as bubbles in reverse time, but have their own distinctive characteristics and dynamics.

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